

SPDEs with coloured noise: Analytic and stochastic approaches

by

MARCO FERRANTE^(*)
Dipartimento di Matematica
Università di Padova
Via Belzoni 7
35131 Padova, Italy
e-mail: ferrante@math.unipd.it

and

MARTA SANZ-SOLÉ^(**)
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585
08007 BARCELONA, Spain
e-mail: marta.sanz@ub.edu

Abstract: We study strictly parabolic stochastic partial differential equations on \mathbb{R}^d , $d \geq 1$, driven by a Gaussian noise white in time and coloured in space. Assuming that the coefficients of the differential operator are random, we give sufficient conditions on the correlation of the noise ensuring Hölder continuity for the trajectories of the solution of the equation. For self-adjoint operators with deterministic coefficients, the mild and weak formulation of the equation are related, deriving path properties of the solution to a parabolic Cauchy problem in evolution form.

Key words: Stochastic partial differential equations, mild and weak solutions, random noise.

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1 Introduction

Stochastic partial differential equations (SPDEs) can be analyzed by different approaches related with the classical deterministic methods. Let us mention the variational point of view ([11], [18], [19]) and the semigroup approach ([5]), based on analytical methods, and the more genuine probabilistic setting using stochastic integration with respect to martingale measures ([26], [4]).

The variational approach leads in particular to a now very complete L_2 -theory (see [19]). However, this theory does not provide with sharp results on the properties of the trajectories of the solutions of SPDEs, except in the time variable. A more deep analytical insight into parabolic SPDEs has been recently given by Krylov and Lototsky, developing an L_p -theory with $p \in [2, \infty)$ (see [12], [13] and the references herein, [14]). This theory allows to obtain properties of the trajectories -both in time and space- quite sharp, using Sobolev type imbeddings. Let us point out that in [12], [13] the coefficients of the differential operator can be random, therefore the theory applies to a very general class of equations. In a similar spirit, parabolic SPDEs with deterministic coefficients in Hölder classes have been studied in [17].

In this paper we study stochastic partial differential equations in the whole space \mathbb{R}^d , with arbitrary dimension $d \geq 1$, driven by a Gaussian noise white in time and with homogeneous spatial correlation. The differential operator is strictly parabolic with random coefficients, the free terms are random as well. Using the analytical approach of [13] (see also [12]), we give sufficient conditions on the correlation of the noise ensuring the existence of a solution with values on some subspace of $L_p(\mathbb{R}^d)$, $p \in [2, \infty)$, and then, by means of Sobolev type imbeddings, we obtain the existence of a random field, indexed by time and space, which is a version of the solution and has its trajectories jointly Hölder continuous in t, x .

A similar question using the evolution approach has been addressed in some previous articles. In fact, parabolic equations with random coefficients in spatial dimension $d = 1$, driven by a space-time white noise have been studied in [2]. The main result is the existence of a *continuous* random field solution to the equation. The mild form of the equation contains a stochastic convolution with an anticipating integrand. Therefore, the analysis requires tools of anticipating stochastic calculus -a intricate machinery based on Malliavin calculus- and needs a strong regularity in terms of the random component ω . These type of hypothesis can be avoided with the analytic approach. In fact, in this situation stronger results are given in [13], Theorem 8.5 and Remark 8.7, where joint *Hölder continuity* is obtained.

For $d \geq 1$ and driving noise of the same kind that the one we are considering in this paper, joint Hölder continuity for the stochastic heat equation in its mild form has been obtained in [22]. The result, proved by means of

Kolmogorov's continuity criterium, is an extension of the one stated in [26] for $d = 1$.

The analytic approach considers the formal SPDE in a *weak* form (see (6), (7)). Studying the relationship between the *weak* and the *mild* formulation of the SPDE (see (24)) gives the possibility of transferring results obtained in the analytic setting to the evolution scenary. The last part of the article is devoted to this topic, in the particular case where the differential operator is self-adjoint and its coefficients are deterministic. In the framework of a L_2 -theory, for a Neumann boundary-value problem with a strictly parabolic divergence operator, this question has been studied in [23].

2 Some preliminaries and notation

We denote by $\mathcal{D}(\mathbb{R}^{d+1})$ the space of Schwartz test functions ([20], page 24). On a complete probability space (Ω, \mathcal{F}, P) , we consider a Gaussian process $\{F(\phi), \phi \in \mathcal{D}(\mathbb{R}^{d+1})\}$, mean zero, with covariance functional given by

$$\begin{aligned} E(F(\phi), F(\psi)) &= \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx) (\phi(s, \cdot) * \tilde{\psi}(s, \cdot))(x) \\ &= \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\phi(s, \cdot)(\xi) \overline{\mathcal{F}\psi(s, \cdot)(\xi)}. \end{aligned} \tag{1}$$

In (1), Γ is a non-negative, non-negative definite, tempered measure, $\tilde{\psi}(s, x) = \psi(s, -x)$, μ is the non-negative tempered measure on \mathbb{R}^d defined by $\mathcal{F}^{-1}\Gamma$, where \mathcal{F} denotes the Fourier transform operator. We notice that Γ is a symmetric measure ([20], Chap. VII, Theorem XVII). For any test function $f, g \in \mathcal{D}(\mathbb{R}^{d+1})$, the functional

$$Q(f, g) = \int_{\mathbb{R}^d} \Gamma(dx) (f * \tilde{g})(x)$$

is non-negative and translation invariant, that means, $Q(f, g) = Q(\tau_x f, \tau_x g)$, where $\tau_x f(\cdot) = f(\cdot + x)$ (see Gel'fand and Vilenkin [8], pag. 169).

Following Dalang and Frangos [3] (see also Dalang [4]) the process F can be extended to a worthy martingale measure in the sense of Walsh. We will denote by $\{F(t, A), t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ this extension and by \mathcal{F}_t the σ -field generated by $\{F(s, A), 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d)\}$.

Consider the inner product on $\mathcal{D}(\mathbb{R}^d)$ defined by

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} \Gamma(dx) (f * \tilde{g})(x).$$

Let \mathcal{H} be the completion of $\mathcal{D}(\mathbb{R}^{d+1})$ with respect to the norm derived from $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. For any complete orthonormal system (CONS) $\{e_j, j \geq 0\} \subset$

$\mathcal{D}(\mathbb{R}^{d+1})$ of \mathcal{H} , define

$$W^k(t) = \int_0^t \int_{\mathbb{R}^d} F(ds, dx) e_k(x), \quad (2)$$

$k \geq 0$, where the integral must be understood in Walsh's sense. The process $\{W^k(t), t \in [0, T], k \geq 0\}$ is a sequence of independent standard Brownian motions.

One can check that for any predictable process X ,

$$\int_0^t \int_{\mathbb{R}^d} F(ds, dx) X(s, x) = \sum_{k=0}^{\infty} \int_0^t W^k(ds) \langle X(s, \cdot), e_k(\cdot) \rangle_{\mathcal{H}}. \quad (3)$$

In particular, for any $\phi \in \mathcal{D}(\mathbb{R}^{d+1})$

$$F(t, \phi) := \int_0^t \int_{\mathbb{R}^d} F(ds, dx) \phi(x) = \sum_{k=0}^{\infty} \langle \phi, e_k \rangle_{\mathcal{H}} W^k(t). \quad (4)$$

Let $p \in (1, +\infty)$, $n \in \mathbb{R}$ and $d \in \mathbb{N}$. We denote by $H_p^n = H_p^n(\mathbb{R}^d)$ the fractional Sobolev space consisting of distributions g on \mathbb{R}^d such that there exists $f \in L_p(\mathbb{R}^d)$ and $g = (1 - \Delta)^{-\frac{n}{2}} f$. It is a Banach space endowed with the norm

$$\|u\|_{n,p} = \|(1 - \Delta)^{n/2} u\|_p,$$

where $\|\cdot\|_p$ denotes the usual norm of $L_p(\mathbb{R}^d)$ and Δ is the Laplacian operator on \mathbb{R}^d . It is important to notice that $\|\cdot\|_{n,p} \leq \|\cdot\|_{m,p}$ for $n \leq m$; this gives rise to the embeddings

$$\cdots \subset H_p^m \subset H_p^n \subset \cdots \subset L_p \subset \cdots \subset H_p^{-n} \subset H_p^{-m} \subset \cdots$$

When $n \in \mathbb{Z}_+$, the spaces H_p^n coincide with the classical Sobolev spaces W_p^n . Moreover, the space \mathcal{C}_0^∞ of infinitely differentiable functions with compact support is dense in each H_p^n . We refer the reader to [1] and [25] for an extensive account on these spaces.

3 SPDEs with random coefficients

In this section, we analyze a parabolic spde, with Lipschitz coefficients, driven by a noise F as has been described in Section 2, under the perspective of the general theory developed in [12], [13]. More precisely, we exhibit a relationship between the covariance measure Γ and a fractional differentiability degree η leading, a.s., to jointly continuous solutions in time and in space.

The results might be considered as a complement of those in Section 8.3 in [13], where the spatial dimension is $d = 1$ and the driving noise, white in

time and in space. Their proof consists in showing that the assumptions of Theorem 5.1 in [13] (see also Theorem 3.6 in [12]) are satisfied.

For the sake of completeness, we start by quoting some basic material from [12], [13].

Consider a fractional Sobolev space H_p^n , with fixed $p \in [2, \infty)$, $n \in \mathbb{R}$. For any $u \in H_p^n$, $\phi \in \mathcal{C}_0^\infty$, we define

$$(u, \phi) = \int_{\mathbb{R}^d} [(1 - \Delta)^{n/2} u](x) [(1 - \Delta)^{-n/2} \phi](x) dx. \quad (5)$$

Let τ be a stopping time with respect to $(\mathcal{F}_t)_{t \geq 0}$ and \mathcal{P} be the predictable σ -field. Set $\mathbb{H}_p^n(\tau) = L_p(\llbracket 0, \tau \rrbracket, \mathcal{P}, H_p^n)$, $\mathbb{H}_p^n := \mathbb{H}_p^n(\infty)$. The spaces $\mathbb{H}_p^n(\tau)$ are a kind of stochastic fractional Sobolev spaces.

We also introduce the following notation:

$$(f, g) \in \mathcal{F}_p^n(\tau) \text{ if and only if } f \in \mathbb{H}_p^n(\tau), g \in \mathbb{H}_p^{n+1}(\tau, l^2), \text{ and we set } \|(f, g)\|_{\mathcal{F}_p^n(\tau)} = \|f\|_{\mathbb{H}_p^n(\tau)} + \|g\|_{\mathbb{H}_p^{n+1}(\tau, l^2)}$$

where $\mathbb{H}_p^{n+1}(\tau, l^2)$ correspond to the space of square summable sequences of elements of $\mathbb{H}_p^{n+1}(\tau)$. We denote by $(w_k(t), t \in [0, T], k \geq 0)$ a sequence of independent standard Wiener processes.

Definition 1 (Definition 3.1, [13]) For a distribution valued function $u \in \cap_{T>0} \mathbb{H}_p^n(\tau \wedge T)$, we write $u \in \mathcal{H}_p^n(\tau)$ if

$$u_{xx} \in \mathbb{H}_p^{n-2}(\tau), u(0, \cdot) \in L_p(\Omega, \mathcal{F}_0, H_p^{n-2/p})$$

and there exists $(f, g) \in \mathcal{F}_p^{n-2}(\tau)$ such that, for any $\phi \in \mathcal{C}_0^\infty$, the equality

$$(u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t ds (f(s, \cdot), \phi) + \sum_{k=1}^{\infty} \int_0^t w^k(ds) (g^k(s, \cdot), \phi)$$

holds for all τ a.s.

We set

$$\|u\|_{\mathcal{H}_p^n(\tau)} = \|u_{xx}\|_{\mathbb{H}_p^{n-2}(\tau)} + \|(f, g)\|_{\mathcal{F}_p^{n-2}(\tau)} + (E\|u(0, \cdot)\|_{n-2/p, p}^p)^{1/p}.$$

Let us recall the result on existence and uniqueness of solution for stochastic partial differential equations of parabolic type driven by a sequence of independent Wiener processes. First, we introduce some notation, then the assumptions and finally, the statement.

Fix $n \in \mathbb{R}$ and $\gamma \in [0, 1[$ be such that $\gamma = 0$ if $n = 0, \pm 1, \pm 2, \dots$; otherwise, $\gamma > 0$ and is such that $|n| + \gamma$ is not an integer. Define

$$B^{|n|+\gamma} = \begin{cases} B(\mathbb{R}^d) & \text{if } n = 0 \\ \mathcal{C}^{|n|-1, 1}(\mathbb{R}^d) & \text{if } n = \pm 1, \pm 2, \dots \\ \mathcal{C}^{|n|+\gamma}(\mathbb{R}^d) & \text{otherwise} \end{cases},$$

where $B(\mathbb{R}^d)$ is the Banach space of bounded functions on \mathbb{R}^d , $\mathcal{C}^{|n|-1,1}(\mathbb{R}^d)$ is the Banach space of $|n| - 1$ times continuously differentiable functions whose derivatives of $(|n| - 1)$ -st order are Lipschitz; $\mathcal{C}^{|n|+\gamma}(\mathbb{R}^d)$ are Hölder spaces. The spaces $B^{|n|+\gamma}(l_2)$ are defined in the obvious way. Consider the following equation on $\llbracket 0, \tau \rrbracket$:

$$\begin{aligned} du(t, x) = & \left[a^{i,j}(t, x) u_{x^i, x^j}(t, x) + f(t, x, u) \right] dt \\ & + g^k(t, x, u) dw_t^k. \end{aligned} \quad (6)$$

Notice that, in comparison with equation (5.1) in Krylov [13], we take here $\sigma^{ik} \equiv 0$.

By a solution to the Cauchy problem for equation (6) with initial condition u_0 , we mean a stochastic process $u \in \mathcal{H}_p^{n+2}(\tau)$ such that for any test function $\phi \in \mathcal{C}_0^\infty$,

$$\begin{aligned} (u(t, \cdot), \phi) = & (u(0, \cdot), \phi) + \int_0^t ds (a^{i,j}(s, \cdot) u_{x^i, x^j}(s, \cdot) + f(s, \cdot, u), \phi) \\ & + \int_0^t w^k(ds) (g^k(s, \cdot, u), \phi), \end{aligned} \quad (7)$$

for all $t \in \llbracket 0, \tau \rrbracket$.

Assume the following conditions on the differential operator and on the coefficients of the equation:

(A1) For any $i, j = 1, \dots, n$,

$$a^{i,j} : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$$

is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. For any $\omega \in \Omega$ a.s. and $t \geq 0$, we have $a^{i,j}(t, \cdot) \in B^{|n|+\gamma}$ and $\|a^{i,j}(t, \cdot)\|_{B^{|n|+\gamma}} \leq K$. Moreover, there exist $K, \delta > 0$, such that for any $\omega \in \Omega$, $t \geq 0$, $x, \lambda \in \mathbb{R}^d$,

$$\delta |\lambda|^2 \leq a^{i,j}(t, x) \lambda^i \lambda^j \leq K |\lambda|^2.$$

(A2) For any $u \in H_p^{n+2}$, $f(t, \cdot, u)$, $g(t, \cdot, u)$ are predictable processes taking values in H_p^n and $H_p^{n+1}(l_2)$, respectively.

In addition,

1. $(f(\cdot, *, 0), g(\cdot, *, 0)) \in \mathcal{F}_p^n(\tau)$,
2. f, g are a.s. continuous in the third variable u ;
3. for any $\varepsilon > 0$, there exists K_ε such that for any $u, v \in H_p^{n+2}$, $t \geq 0$,

$$\begin{aligned} & \|f(t, \cdot, u) - f(t, \cdot, v)\|_{n,p} + \|g(t, \cdot, u) - g(t, \cdot, v)\|_{n+1,p} \\ & \leq \varepsilon \|u - v\|_{n+2,p} + K_\varepsilon \|u - v\|_{n,p}, \end{aligned}$$

a.s.

The next result is a particular version of Theorem 5.1 in [13]

Theorem 2 *Let $u_0 \in L_p(\Omega, \mathcal{F}_0, H_p^{n+2-2/p})$. Then the Cauchy problem (6) on $\llbracket 0, \tau \rrbracket$ with initial condition $u(0, \cdot) = u_0$ has a unique solution $u \in \mathcal{H}_p^{n+2}(\tau)$. This solution satisfies*

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{n+2}(\tau)} &\leq N \left\{ \|f(\cdot, *, 0)\|_{\mathbb{H}_p^n(\tau)} + \|g(\cdot, *, 0)\|_{\mathbb{H}_p^{n+1}(\tau, l_2)} \right. \\ &\quad \left. + (E\|u_0\|_{n+2-2/p, p}^p)^{1/p} \right\}, \end{aligned}$$

where the constant N depends only on $d, n, \gamma, p, \delta, K, T$ and the function K_ε .

Consider now the equation

$$\begin{aligned} du(t, x) &= [a^{i,j}(t, x)u_{x^i, x^j}(t, x) + b^i(t, x)u_{x^i}(t, x) \\ &\quad + f(t, x, u(t, x))]dt + h(t, x, u(t, x))F(dt, x), \end{aligned} \quad (8)$$

with initial condition $u(0, x) = u_0(x)$, where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$ and F is the Gaussian process introduced in the preceding section. The coefficients f, h are random real functions defined on $\llbracket 0, \tau \rrbracket \times \mathbb{R}^d \times \mathbb{R}$. Under suitable assumptions, we shall prove that this equation can be set in the framework of Theorem 2 and deduce Hölder continuity of the trajectories of its unique solution.

Let us write (8) into the form (6). We consider a CONS $\{e_j, j \geq 0\}$ of \mathcal{H} . We have,

$$\begin{aligned} \langle h(t, \cdot, u), e_k \rangle_{\mathcal{H}} &= \int_{\mathbb{R}^d} \Gamma(dx) (h(t, \cdot, u) * \tilde{e}_k)(x) \\ &= \int_{\mathbb{R}^d} dy h(t, y, u) \int_{\mathbb{R}^d} \Gamma(dx) \tilde{e}_k(x - y), \end{aligned} \quad (9)$$

where in the last equality we have applied Fubini's theorem.

Therefore, the term $h(t, x, u(t, x))F(dt, x)$ can be rewritten as $g^k(t, x, u(t, x))W^k(dt)$ with

$$g^k(t, x, u(t, x)) = h(t, x, u(t, x)) \int_{\mathbb{R}^d} \Gamma(dy) \tilde{e}_k(y - x),$$

and W^k defined in (2) (see (3)).

Indeed, in the integral formulation, the contribution of the last term in (8) is, for $\phi \in \mathcal{C}_0^\infty$, $\int_0^t \int_{\mathbb{R}^d} F(dt, dx) \phi(x) h(t, x, u(t, x))$. By virtue of (3) and (9),

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} F(dt, dx) \phi(x) h(t, x, u(t, x)) &= \sum_{k=0}^{\infty} \int_0^t W^k(ds) \langle h(t, \cdot, u(t, \cdot)) \phi, e_k \rangle_{\mathcal{H}} \\ &= \sum_{k=0}^{\infty} \int_0^t W^k(ds) \int_{\mathbb{R}^d} h(t, y, u(t, y)) \phi(y) \left(\int_{\mathbb{R}^d} \Gamma(dx) \tilde{e}_k(x - y) \right) \\ &= \sum_{k=0}^{\infty} \int_0^t W^k(ds) \left(h(t, \cdot, u(t, \cdot)) \int_{\mathbb{R}^d} \Gamma(dx) \tilde{e}_k(x - \cdot), \phi \right)_2, \end{aligned}$$

where $(\cdot, \cdot)_2$ denotes the inner product in $L^2(\mathbb{R}^d)$.

Set

$$v_k(x) = \int_{\mathbb{R}^d} \Gamma(dy) \tilde{e}_k(y - x).$$

The following lemma provides a useful tool to apply Theorem 2 to equation (8), where $h(t, x, u(t, x))F(dt, x)$ is replaced by $g^k(t, x, u(x, t))W^k(dt)$. For any $\eta > 0$, we denote by $R_{\eta, d}(x)$ the kernel of the operator $(1 - \Delta)^{-\eta/2}$ on \mathbb{R}^d , that is,

$$\left[(1 - \Delta)^{-\eta/2} u \right] (x) = R_{\eta, d} * u.$$

It is well known that

$$R_{\eta, d}(x) = C_{\eta, d} |x|^{\frac{\eta-d}{2}} K_{\frac{d-\eta}{2}}(|x|),$$

where $C_{\eta, d}$ is the reciprocal of $\pi^{d/2} 2^{(d+\eta-2)/2} \Gamma(\frac{\eta}{2})$ and K_ν is the modified Bessel function of the third kind (see [6]). Notice that $R_{\eta, d}(x)$ is a radial function and that

$$R_{\eta_1, d} * R_{\eta_2, d} = R_{\eta_1 + \eta_2, d}$$

for any $\eta_1, \eta_2 > 0$. Hence,

$$\nu_{\eta, d} := \|R_{\eta, d}\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^d} \Gamma(dx) R_{2\eta, d}(x). \quad (10)$$

Lemma 3 *Let $\eta \in (0, \infty)$, $d \in \mathbb{N}$ be such that*

$$\nu_{\eta, d} = \|R_{\eta, d}\|_{\mathcal{H}}^2 < \infty. \quad (11)$$

Let $h \in L_p(\mathbb{R}^d)$, $g^k = v_k h$. Then $g = \{g_k, k \geq 0\} \in H_p^{-\eta}(l_2)$ and

$$\|g\|_{-\eta, p} = \|\bar{h}\|_p \leq C \|h\|_p, \quad (12)$$

with

$$\bar{h}(x) = \|R_{\eta, d}(x - \cdot)h\|_{\mathcal{H}}$$

and $C = \nu_{\eta, d}^{1/2}$.

Proof. Fubini's theorem and Parseval's identity yield

$$\begin{aligned} \|(1 - \Delta)^{-\eta/2} g(x)\|_{l_2}^2 &= \sum_{k=0}^{\infty} \left((1 - \Delta)^{-\eta/2} g^k(x) \right)^2 \\ &= \sum_{k=0}^{\infty} \left((R_{\eta, d} * (v_k h))(x) \right)^2 \\ &= \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^d} dy R_{\eta, d}(x - y) \left(\int_{\mathbb{R}^d} \Gamma(dz) \tilde{e}_k(z - y) h(y) \right) \right)^2 \\ &= \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^d} \Gamma(dz) \left(\int_{\mathbb{R}^d} dy R_{\eta, d}(x - y) h(y) \tilde{e}_k(z - y) \right) \right)^2 \\ &= \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^d} \Gamma(dz) (R_{\eta, d}(x - \cdot) h * \tilde{e}_k)(z) \right)^2 \\ &= \sum_{k=0}^{\infty} \langle R_{\eta, d}(x - \cdot) h, e_k \rangle_{\mathcal{H}}^2 = \|R_{\eta, d}(x - \cdot) h\|_{\mathcal{H}}^2. \end{aligned}$$

Therefore,

$$\begin{aligned}\|g\|_{-\eta,p} &= \|(1 - \Delta)^{-\eta/2} g\|_{L_p(l_2)} = \left(\int_{\mathbb{R}^d} dx \|(1 - \Delta)^{-\eta/2} g(x)\|_{l_2}^p \right)^{1/p} \\ &= \left(\int_{\mathbb{R}^d} dx \|R_{\eta,d}(x - \cdot) h\|_{\mathcal{H}}^p \right)^{1/p} = \|\bar{h}\|_p.\end{aligned}$$

The second part of (12) is a consequence of Hölder's inequality. Indeed, first we notice that, since Γ is translation invariant,

$$\nu_{\eta,d} = \|R_{\eta,d}\|_{\mathcal{H}}^2 = \|R_{\eta,d}(x - \cdot)\|_{\mathcal{H}}^2. \quad (13)$$

Then,

$$\begin{aligned}\|\bar{h}\|_p^p &= \int_{\mathbb{R}^d} dx \|R_{\eta,d}(x - \cdot) h\|_{\mathcal{H}}^p \\ &= \int_{\mathbb{R}^d} dx \left(\int_{\mathbb{R}^d} \Gamma(dy) \int_{\mathbb{R}^d} dz R_{\eta,d}(x - (y - z)) h(y - z) \tilde{R}_{\eta,d}(x - z) \tilde{h}(z) \right)^{\frac{p}{2}} \\ &\leq \int_{\mathbb{R}^d} dx (\|R_{\eta,d}(x - \cdot)\|_{\mathcal{H}}^{\frac{p}{2}-1}) \\ &\quad \times \left[\int_{\mathbb{R}^d} \Gamma(dy) \int_{\mathbb{R}^d} dz R_{\eta,d}(x - (y - z)) \tilde{R}_{\eta,d}(x - z) |h(y - z) \tilde{h}(z)|^{\frac{p}{2}} \right].\end{aligned}$$

Thus, (13), Fubini's theorem and Schwarz's inequality and the invariance of Lebesgue measure imply

$$\begin{aligned}\|\bar{h}\|_p^p &\leq \nu_{\eta,d}^{\frac{p}{2}-1} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} \Gamma(dy) R_{\eta,d}(y - z) R_{\eta,d}(z) \\ &\quad \times |h(y - z + x)|^{\frac{p}{2}} |\tilde{h}(z + x)|^{\frac{p}{2}} \\ &\leq \nu_{\eta,d}^{\frac{p}{2}-1} \int_{\mathbb{R}^d} \Gamma(dy) \left(R_{\eta,d} * \tilde{R}_{\eta,d} \right)(y) \left(\int_{\mathbb{R}^d} dx |h(y - z + x)|^p \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{\mathbb{R}^d} dx |\tilde{h}(z + x)|^p \right)^{\frac{1}{2}} \\ &= \nu_{\eta,d}^{\frac{p}{2}} \|h\|_p^p.\end{aligned}$$

This completes the proof of the lemma. ■

Remark 4 Let $\Gamma(dx) = \delta_{\{0\}}(x)$ and thus, $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_2$. In this particular case (12) has been obtained in Lemma 8.4 of Krylov [13].

Proposition 4.4.1 in [16] establishes that, if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^2)^\eta} < +\infty.$$

then (11) holds true.

The behavior of the Bessel function K_ν is well-known (see for instance Donoghue [6]). In fact, in a neighborhood O^+ of 0,

$$K_\nu(r) \sim \begin{cases} \log(r), & \text{if } \nu = 0, \\ r^{-|\nu|}, & \text{if } \nu \neq 0. \end{cases}$$

While away from zero,

$$K_\nu(r) = C_\nu e^{-r}.$$

This leads to the following conclusions, which have already appeared in previous discussions on different classes of spde's (for instance, in [21], [16]).

1. Assume $0 < \eta < \frac{d}{2}$. Then

$$\nu_{\eta,d} < +\infty \Leftrightarrow \int_{O^+} |x|^{2\eta-d} \Gamma(dx) < +\infty.$$

2. Let $\eta = \frac{d}{2}$. Then

$$\nu_{\eta,d} < +\infty \Leftrightarrow \int_{O^+} |x|^{\frac{2\eta-d}{2}} \log\left(\frac{1}{|x|}\right) \Gamma(dx) < +\infty.$$

3. If $\eta > \frac{d}{2}$. Then $\nu_{\eta,d} < +\infty$, without any additional condition on Γ .

Example 5 (Riesz kernels): Set $\Gamma(dx) = |x|^{-\alpha} dx$, with $\alpha \in (0, d)$. Then, for $\eta \in (0, \frac{d}{2}]$, $\nu_{\eta,d} < +\infty$ if and only if $\alpha \in (0, 2\eta \wedge d)$.

Let us now introduce the set of hypotheses to be assumed in order to prove existence and uniqueness of solution for (8) and Hölder properties for its paths. Given $\gamma_1, \gamma_2 > 0$, we denote by $\mathcal{C}^{\gamma_1, \gamma_2}([0, t] \times \mathbb{R}^d)$, the space of real-valued functions defined on $[0, t] \times \mathbb{R}^d$, jointly Hölder continuous of order γ_1 in its first variable and γ_2 in its second one.

(H1) For any $i, j = 1, \dots, n$, $a^{i,j}, b^i : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable such that, for any $\omega \in \Omega$ a.s. and $t \leq 0$, $a^{i,j}(\omega, t, \cdot) \in \mathcal{C}^\alpha(\mathbb{R}^d)$, $\alpha \in (\frac{3}{2}, 2)$, $b^i(\omega, t, \cdot) \in \mathcal{C}^{0,1}(\mathbb{R}^d)$, and

$$\sup_{t \leq 0} [\|a(t, \cdot)\|_{\mathcal{C}^\alpha} + \|b(t, \cdot)\|_{\mathcal{C}^{0,1}}] \leq k.$$

There exist $K, \delta > 0$, such that for any $\omega \in \Omega$ a.s., $t \geq 0$, $x, \lambda \in \mathbb{R}^d$,

$$\delta |\lambda|^2 \leq a^{i,j}(t, x) \lambda^i \lambda^j \leq K |\lambda|^2.$$

(H2) $f, h : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ are such that, for any x and u , $f(\cdot, x, u)$, $h(\cdot, x, u)$ are predictable and

$$\sup_{(\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^d} [|f(t, x, u) - f(t, x, v)| + |h(t, x, u) - h(t, x, v)|] \leq k |u - v|,$$

for some positive constant k , a.s.

We recall that, for $\alpha \in (\frac{3}{2}, 2)$, \mathcal{C}^α is the space of continuously differentiable functions whose partial derivatives of first order are $\{\alpha\}$ - Hölder continuous, where $\alpha = [\alpha] + \{\alpha\}$, $[\alpha]$ meaning the integer part of α (see [25]); $\mathcal{C}^{0,1}$ is the space of Lipschitz continuous functions.

In the proof of the next theorem we will use the following Remark 5.5 of [13]:

For any $u \in H_p^{n+2}$, $m \in [n, n+2]$ and $\varepsilon > 0$, we have

$$\begin{aligned} \|u\|_{m,p} &\leq N \|u\|_{n+2,p}^\theta \|u\|_{n,p}^{1-\theta} \\ &\leq N\theta\varepsilon \|u\|_{n+2,p} + N(1-\theta)\varepsilon^{-\frac{\theta}{1-\theta}} \|u\|_{n,p}, \end{aligned}$$

where $\theta = \frac{m-n}{2}$ and N depends only on d, n, m and p .

In the following theorem τ denotes a fixed stopping time with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ defined in Section 2.

Theorem 6 *Suppose that there exists $\eta \in (\frac{1}{2}, 1)$ such that*

$$\nu_{\eta,d} = \|R_{\eta,d}\|_{\mathcal{H}}^2 < +\infty.$$

We also assume that, for some $p \in [2, +\infty)$ the following conditions are satisfied:

(a) $u_0 \in L_p(\Omega, \mathcal{F}_0, H_p^{1-\eta-\frac{2}{p}}),$

(b)

$$I^p(\tau) = E \left[\int_0^\tau dt \left(\|f(t, \cdot, 0)\|_{-1-\eta,p}^p + \|\bar{h}(t, \cdot, 0)\|_p^p \right) \right] < +\infty, \quad (14)$$

where

$$\bar{h}(t, x, 0) := \|R_{\eta,d}(x - \cdot)h(t, \cdot, 0)\|_{\mathcal{H}}. \quad (15)$$

Then, in the space $\mathcal{H}_p^{1-\eta}(\tau)$, equation (8) with the initial condition u_0 and coefficients satisfying (H1), (H2) possesses a unique solution u . Moreover,

$$\|u\|_{\mathcal{H}_p^{1-\eta}(\tau)} \leq C \left(I(\tau) + \left(E(\|u_0\|_{1-\eta-\frac{2}{p}}^p)^{\frac{1}{p}} \right) \right), \quad (16)$$

where the constant C depends on $\eta, d, \alpha, p, \delta, k$ and τ .

In addition, if conditions (a), (b) are satisfied for any $p \geq 2$ then, the trajectories of u belong to the space of Hölder continuous functions $\mathcal{C}^{\gamma_1, \gamma_2}([0, \tau] \times \mathbb{R}^d)$, a.s. with $\gamma_1 \in (0, \frac{1-\eta}{2})$, $\gamma_2 \in (0, 1-\eta)$.

Proof.

The existence and uniqueness of solution will follow by applying Theorem 2 to

$$\begin{aligned} f(t, x, u) &:= b^i(t, x)u_{x^i}(t, x) + f(t, x, u(t, x)), \\ g^k(t, x, u) &:= h(t, x, u(t, x))v_k(x), \end{aligned}$$

and by taking $n = -(1 + \eta)$. In fact, we will check that the hypotheses (A1) and (A2) are satisfied.

Since $n \in (-2, -\frac{3}{2})$, we shall consider as space $B^{|n|+\gamma}$, with $\gamma > 0$ and $|n| + \gamma$ not an integer, the space $\mathcal{C}^\alpha(\mathbb{R}^d)$, with $\alpha \in (\frac{3}{2}, 2)$.

Set $\bar{f}(t, x, u) = b^i(t, x)u_{x^i}(t, x) + f(t, x, u(t, x))$; we have to check the following conditions for $n = -(1 + \eta)$ (see Assumption **(A2)** before):

- (1) For any $u \in H_p^{n+2}$, $\{\bar{f}(t, \cdot, u), t \geq 0\}$ is a predictable process with values on H_p^n .
- (2) $\bar{f}(\cdot, \cdot, 0) \in \mathbb{H}_p^n$, a.s.
- (3) \bar{f} is a continuous function in u a.s.
- (4) For any $\varepsilon > 0$, there exists K_ε such that, for every $u, v \in H_p^{n+2}, t, \omega$,

$$\|\bar{f}(t, \cdot, u) - \bar{f}(t, \cdot, v)\|_{n,p} \leq \varepsilon \|u - v\|_{n+2,p} + K_\varepsilon \|u - v\|_{n,p}.$$

The predictability of \bar{f} clearly follows from the same property of b and f . Let $u \in H_p^{n+2}$; then $u_{x^i} \in H_p^{n+1}$. Notice that, since $|n+1| \in (\frac{1}{2}, 1)$, the space $B^{|n+1|+\gamma}$ coincides with the space of the α -Hölder continuous functions for some $\alpha \in (0, 1)$. Since $\mathcal{C}^{0,1}(\mathbb{R}^d) \subset \mathcal{C}^\alpha(\mathbb{R}^d)$, Lemma 5.2 in Krylov [13] applied to b and u yields

$$\|b^i u_{x^i}\|_{n+1,p} \leq \|b\|_{B^{|n+1|+\gamma}} \|u_{x^i}\|_{n+1,p} < +\infty. \quad (17)$$

Thus $b^i u_{x^i} \in H_p^n$.

Moreover, $u \in L_p$ and the Lipschitz condition of f with respect to u implies $f(u) - f(0) \in L_p \subset H_p^n$. By (14), $f(t, \cdot, 0) \in H_p^n$ a.e. on $\llbracket 0, \tau \rrbracket$. Therefore, (2) holds. In addition

$$|f(t, \cdot, u)| \leq k|u| + |f(t, \cdot, 0)|.$$

This proves $f(t, \cdot, u) \in H_p^n$ and thus $\bar{f}(t, \cdot, u) \in H_p^n$ as well. Since $n < -1$, applying again Lemma 5.2 in [13], we have

$$\begin{aligned} \|b^i u_{x^i}\|_{n,p} &\leq \|b^i u_{x^i}\|_{-1,p} \leq \|b\|_{\mathcal{C}^{0,1}} \|u_{x^i}\|_{-1,p} \\ &\leq K \|u\|_p = K \|u\|_{n+2+\eta-1,p}, \end{aligned} \quad (18)$$

where in the last identity we have used that $n + 2 + \eta - 1 = 0$. This fact, together with the Lipschitz property of f with respect to u , prove **(3)**.

We also have

$$\|f(t, \cdot, u) - f(t, \cdot, v)\|_{n,p} \leq \|f(t, \cdot, u) - f(t, \cdot, v)\|_p \leq K\|u - v\|_p. \quad (19)$$

Then

$$\|\bar{f}(t, \cdot, u) - \bar{f}(t, \cdot, v)\|_{n,p} \leq A + B,$$

where

$$A = \|f(t, \cdot, u) - f(t, \cdot, v)\|_{n,p} \leq K\|u - v\|_p,$$

$$B = \|b^i u_{x^i} - b^i v_{x^i}\|_{n,p} \leq K\|u - v\|_p,$$

by (19) and (18), respectively.

We now apply the above quoted Remark 5.5 of Krylov [13] to $m = n + 2 + \eta - 1 = 0$. Notice that $\theta = -\frac{n}{2} > 0$, $1 - \theta = \frac{2+n}{2} > 0$, $-\frac{\theta}{1-\theta} = \frac{n}{2+n} < 0$. This yields property **(4)**.

Concerning the coefficient $g(t, x, u) = \{h(t, x, u(t, x)v_k(x))\}_{k \geq 0}$ we have to check first of all that, for any $u \in H_p^{n+2}$, $\{g(t, \cdot, u), t \geq 0\}$ is a predictable process with values on $H_p^{n+1}(l_2)$. This is a simple consequence of the fact that h is predictable and $v_k(x)$ is deterministic. Moreover, since h is Lipschitz, it is immediate also to prove that g is a.s. continuous in u .

Let us now prove that $g(\cdot, *, 0) \in \mathbb{H}_p^{n+1}(\tau, l_2)$. Since $n + 1 = -\eta$, Lemma 3 yields

$$\begin{aligned} \|g(t, \cdot, u)\|_{n+1,p} &\leq \|g(t, \cdot, u) - g(t, \cdot, 0)\|_{n+1,p} + \|g(t, \cdot, 0)\|_{n+1,p} \\ &\leq \|h(t, \cdot, u) - h(t, \cdot, 0)\|_p + \|h(t, \cdot, 0)\|_p \\ &\leq |u| + \|h(t, \cdot, 0)\|_p < +\infty. \end{aligned}$$

and

$$\|g(t, \cdot, 0)\|_{n+1,p} = \|\bar{h}(t, \cdot, 0)\|_p,$$

with $\bar{h}(t, \cdot, 0)$ defined in (15). Thus

$$\begin{aligned} \|g(t, \cdot, 0)\|_{\mathbb{H}_p^{n+1}(\tau, l_2)}^p &= E\left(\int_0^\tau \|g(t, *, 0)\|_{n+1,p}^p dt\right) \\ &= E\left(\int_0^\tau \|\bar{h}(t, *, 0)\|_p^p dt\right) < +\infty, \end{aligned}$$

by (14) and we obtain $g(\cdot, *, 0) \in \mathbb{H}_p^{n+1}(\tau, l_2)$.

It remains to check that for any $\varepsilon > 0$, there exists a constant K_ε such that, for any $u, v \in H_p^{n+2}$, t, ω ,

$$\|g(t, \cdot, u) - g(t, \cdot, v)\|_{n+1,p} \leq \varepsilon\|u - v\|_{n+2,p} + K_\varepsilon\|u - v\|_{n,p}.$$

Applying again Lemma 3 and the Lipschitz property of h , yield

$$\begin{aligned} \|g(t, \cdot, u) - g(t, \cdot, v)\|_{n+1,p} &\leq C \|h(t, \cdot, u) - h(t, \cdot, v)\|_p \\ &\leq K \|u - v\|_p = K \|u - v\|_{n+2+\eta-1,p}. \end{aligned}$$

Then the above property follows, as for \bar{f} , from the above-mentioned Remark 5.5 in [13].

This finishes the proof of the existence and uniqueness of solution for equation (8) in the space $\mathcal{H}_p^{1-\eta}(\tau)$, and of the bound (16).

Let us now check the Hölder continuity of the trajectories of the solution, in a similar way as in Remark 8.7 in [13]. Let $p > 2$, $\frac{1}{2} > \beta > \alpha > \frac{1}{p}$. Then by Theorem 7.2 in [13], a.s.

$$u \in \mathcal{C}^{\alpha-1/p}([0, \tau], H_p^{1-\eta-2\beta}).$$

The space $H_p^{1-\eta-2\beta}$ is embedded into $\mathcal{C}^\gamma(\mathbb{R}^d)$ for $\gamma < 1 - \eta - 2\beta - \frac{d}{p}$, whenever $1 - \eta - 2\beta - \frac{d}{p} > 0$ (see for instance Theorem E.12 of [24]). Thus, taking p big enough and α, β small, we prove that u is γ_2 -Hölder continuous in x , with $\gamma_2 < 1 - \eta$, uniformly in t . On the other hand, the conditions $\beta < \frac{1}{2}$, $1 - \eta - 2\beta - \frac{d}{p} > 0$ are simultaneously satisfied for p big enough whenever $\beta < \frac{1-\eta}{2} \wedge \frac{1}{2} = \frac{1-\eta}{2}$. Thus u is Hölder continuous in t of order $\gamma_1 < \frac{1-\eta}{2}$, uniformly in x .

This finishes the proof of the theorem. ■

Remark 7 *The assumptions of Theorem 6 ensuring Hölder continuity are satisfied if, for instance, $u_0(\omega, \cdot)$ is a.s. a \mathcal{C}^∞ function with compact support and $f(t, x, 0) = h(t, x, 0) = 0$.*

4 Mild formulation: results on the existence and uniqueness of a solution

In this section we consider the formal expression (8), but now, we assume that the coefficients a, b are deterministic. More precisely, we fix a finite time horizon $T > 0$ and we assume the following set of assumptions:

(H1') $a^{i,j}, b^i : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, $i, j = 1, \dots, d$ are $\frac{\alpha}{2}$ -Hölder continuous in $t \in [0, T]$, α -Hölder continuous in $x \in \mathbb{R}^d$, for some $\alpha \in (0, 1)$. In addition, for any $\lambda \in \mathbb{R}^d$, there exist $K, \delta > 0$ such that

$$\delta |\lambda|^2 \leq a^{i,j}(t, x) \lambda^i \lambda^j \leq K |\lambda|^2. \quad (20)$$

(H2') $f, h : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ are such that, for any $x \in \mathbb{R}^d$ and $u \in \mathbb{R}$, $f(\cdot, x, u)$, $h(\cdot, x, u)$ are predictable processes satisfying the Lipschitz condition

$$\begin{aligned} & \sup_{(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d} [|f(t, x, u) - f(t, x, v)| + |h(t, x, u) - h(t, x, v)|] \\ & \leq k|u - v|, \end{aligned}$$

for any $u, v \in \mathbb{R}$.

Following classical approaches on spde's, one can think of equation (8) with the initial condition $u(0, x) = u_0(x)$ as a stochastic Cauchy problem

$$\begin{cases} \mathcal{L}u(t, x) = f(t, x, u(t, x)) + h(t, x, u(t, x))F(dt, dx) \\ u(0, x) = u_0(x) \end{cases} \quad (21)$$

where \mathcal{L} is the second order operator with coefficients depending on t and x , acting on functions defined on $[0, T] \times \mathbb{R}^d$, given by

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{i,j=1}^d a^{i,j}(t, x) \partial_{x_i x_j}^2 - \sum_{i=1}^d b^i(t, x) \partial_{x_i}. \quad (22)$$

By virtue of (20) the operator \mathcal{L} is uniformly parabolic in $[0, T] \times \mathbb{R}^d$ ([15], page 11)).

Let $G(t, x; s, y)$ be the fundamental solution of $\mathcal{L}u = 0$. G is a function defined on $[0, T] \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^d \cap \{(s, t) : 0 \leq s \leq t \leq T\}$. Under the above assumptions on the coefficients of \mathcal{L} , G is continuous in all its variables and for any fixed $s \in [0, T], y \in \mathbb{R}^d$, $G(\cdot, \cdot, s, y)$ is twice continuously differentiable in x , once continuously differentiable in t and satisfies the estimates

$$|\partial_x^\mu \partial_t^\nu G(t, x; s, y)| \leq C(t-s)^{-\frac{d+|\mu|+2\nu}{2}} \exp\left(-c \frac{|x-y|^2}{t-s}\right) \quad (23)$$

where $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}^d, \nu \in \mathbb{N}, |\mu| + 2\nu \leq 2$, with $|\mu| = \sum_{j=1}^d \mu_j$ (see (13.3), page 376 in [15]). Moreover, G is a positive function (Theorem 11 in [7]).

Let us now introduce the notion of *mild solution*. A predictable stochastic process $\{u^M(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ is said to be a mild solution to the stochastic Cauchy problem (21) if it satisfies the equation

$$\begin{aligned} u^M(t, x) &= \int_{\mathbb{R}^d} dy G(t, x; 0, y) u_0(y) \\ &+ \int_0^t \int_{\mathbb{R}^d} F(ds, dy) G(t, x; s, y) h(s, y, u^M(s, y)) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} dy G(t, x; s, y) f(s, y, u^M(s, y)). \end{aligned} \quad (24)$$

Notice that, in order to give a rigorous meaning to equation (24), we must specify the space where the solution belongs to.

Using the CONS $\{e_k, k \geq 0\}$ of \mathcal{H} introduced in Section 2, the stochastic integral in (24) can also be written as

$$\sum_{k=0}^{\infty} \int_0^t W^k(ds) \langle G(t, x; s, \cdot) h(s, \cdot, u^M(s, \cdot)), e_k \rangle,$$

with $W^k(t) = \int_0^t \int_{\mathbb{R}^d} F(ds, dy) e_k(y)$.

In the sequel, we denote by $G_0(t, x)$ the d -dimensional Gaussian density, zero mean, with variance $t \text{Id}_n$. Notice that (23) implies

$$|G(t, x; s, y)| = G(t, x; s, y) \leq C_1 G_0(C_2(t - s), (x - y)),$$

for some positive constants C_1, C_2 .

For any $p \in [2, \infty)$, let \mathcal{B}_p be the Banach space of real-valued predictable processes such that

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} E(|u(t, x)|^p) < \infty.$$

Theorem 8 Fix $p \in [2, \infty)$. Assume $(H1')$, $(H2')$ and

$$\int_0^T ds \sup_{y \in \mathbb{R}^d} E(|h(s, y, 0)|^p + |f(s, y, 0)|^p) < \infty. \quad (25)$$

Suppose also that $\|u_0\|_{\infty} < C$ and moreover,

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty. \quad (26)$$

Then, there exists a unique stochastic processes $\{u^M(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ belonging to \mathcal{B}_p and satisfying (24).

Proof. Following classical ideas (see e.g. [23]), we show that the map

$$\begin{aligned} \mathcal{T}u(t, x) &= \int_{\mathbb{R}^d} dy G(t, x; 0, y) u_0(y) \\ &+ \int_0^t \int_{\mathbb{R}^d} F(ds, dy) G(t, x; s, y) h(s, y, u(s, y)) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} dy G(t, x; s, y) f(s, y, u(s, y)) \end{aligned} \quad (27)$$

possesses a unique fixed point in \mathcal{B}_p .

First, we show that (27) defines a map $\mathcal{T} : \mathcal{B}_p \rightarrow \mathcal{B}_p$. Indeed,

$$E(|\mathcal{T}u(t, x)|^p) \leq C(T_1(t, x) + T_2(t, x) + T_3(t, x)),$$

with

$$\begin{aligned} T_1(t, x) &= \left| \int_{\mathbb{R}^d} dy G(t, x; 0, y) u_0(y) \right|^p, \\ T_2(t, x) &= E \left(\left| \int_0^t \int_{\mathbb{R}^d} F(ds, dy) G(t, x; s, y) h(s, y, u(s, y)) \right|^p \right), \\ T_3(t, x) &= E \left(\left| \int_0^t ds \int_{\mathbb{R}^d} dy G(t, x; s, y) f(s, y, u(s, y)) \right|^p \right). \end{aligned}$$

Clearly,

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} T_1(t, x) \leq \|u_0\|_\infty^p \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \left| \int_{\mathbb{R}^d} G(t, x; 0, y) dy \right|^p < \infty.$$

Notice that

$$\begin{aligned} & \int_{\mathbb{R}^d} \Gamma(dz) (G_0(t-s, x-\cdot) * G_0(t-s, x-\cdot))(z) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_0(s, \cdot)(\xi)|^2 \leq \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty, \end{aligned} \quad (28)$$

which implies

$$\begin{aligned} & \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) (G(t, x; s, \cdot) * \tilde{G}(t, x; s, \cdot))(z) \\ & \leq \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) (G_0(t-s, x-\cdot) * G_0(t-s, x-\cdot))(z) \\ & \leq \int_0^t ds \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \end{aligned}$$

Consequently

$$\sup_{0 \leq t \leq T} \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) (G(t, x; s, \cdot) * \tilde{G}(t, x; s, \cdot))(z) < \infty. \quad (29)$$

Burkholder's and Hölder's inequalities yield

$$\begin{aligned} T_2(t, x) &\leq E \left(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy G(t, x; s, y) h(s, y, u(s, y)) \right. \\ &\quad \left. \times G(t, x; s, y-z) h(s, y-z, u(s, y-z)) \right)^{\frac{p}{2}} \\ &\leq \left(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) (G(t, x; s, \cdot) * \tilde{G}(t, x; s, \cdot))(z) \right)^{\frac{p}{2}-1} \\ &\quad \times \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy G(t, x; s, y) G(t, x; s, y-z) \\ &\quad \times E \left(|h(s, y-z, u(s, y-z))|^{\frac{p}{2}} |h(s, y, u(s, y))|^{\frac{p}{2}} \right). \end{aligned}$$

Hence, by virtue of (28), (29) and the Lipschitz continuity of h , we obtain

$$\begin{aligned} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} T_2(t,x) &\leq C \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \int_0^t ds \sup_y E(|u(s,y)|^p + |h(s,y,0)|^p) \\ &\times \int_{\mathbb{R}^d} \Gamma(dz) G_0(t-s, x-\cdot) * G_0(t-s, x-\cdot) < \infty, \end{aligned}$$

Similarly,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} T_3(t,x) < \infty.$$

Therefore $\mathcal{T}u(t,x)$ belongs to the Banach space \mathcal{B}_p .

To conclude the proof, one can follow the arguments of the proof of Proposition 3 in [23] (see also the end of the proof of the next theorem).

■

In order to compare mild and weak solutions, we need a new version of existence of mild solution.

Theorem 9 Fix $p \in [2, \infty)$. Assume $(H1')$, $(H2')$ and

$$E \left[\int_0^T ds \left(\|h(s, \cdot, 0)\|_p^p + \|f(s, y, 0)\|_p^p \right) \right] < \infty. \quad (30)$$

Suppose also that $u_0 \in L_p(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty.$$

Then, there exists a unique stochastic process $\{u^M(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ that belongs to $L_p(\Omega \times [0,T]; L_p(\mathbb{R}^d))$ and satisfies (24).

Proof. We shall divide the proof into two steps, proving first that the map defined by (27) on $L_p(\Omega \times [0,T]; L_p(\mathbb{R}^d))$, takes values on the same space and secondly, that this map is a contraction.

Step 1: Let $u \in L_p(\Omega \times [0,T]; L_p(\mathbb{R}^d))$; then,

$$E \left(\int_0^T dt \int_{\mathbb{R}^d} dx |\mathcal{T}u(t,x)|^p \right) \leq C(T_1 + T_2 + T_3),$$

with

$$\begin{aligned} T_1 &= \int_0^T dt \int_{\mathbb{R}^d} dx \left| \int_{\mathbb{R}^d} dy G(t,x;0,y) u_0(y) \right|^p, \\ T_2 &= \int_0^T dt \int_{\mathbb{R}^d} dx E \left(\left| \int_0^t \int_{\mathbb{R}^d} F(ds, dy) G(t,x;s,y) h(s,y,u(s,y)) \right|^p \right), \\ T_3 &= \int_0^T dt \int_{\mathbb{R}^d} dx E \left(\left| \int_0^t ds \int_{\mathbb{R}^d} dy G(t,x;s,y) f(s,y,u(s,y)) \right|^p \right). \end{aligned}$$

Hölder's inequality, the properties of G and Fubini's theorem yield

$$\begin{aligned}
T_1 &\leq \int_0^T dt \int_{\mathbb{R}^d} dx \left(\int_{\mathbb{R}^d} dy G(t, x; 0, y) \right)^{p-1} \int_{\mathbb{R}^d} dy G(t, x; 0, y) |u_0(y)|^p \\
&\leq \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \left(\int_{\mathbb{R}^d} dy G(t, x; 0, y) \right)^{p-1} \int_0^T dt \int_{\mathbb{R}^d} dy |u_0(y)|^p \\
&\quad \times \int_{\mathbb{R}^d} dx G(t, x; 0, y) \\
&\leq C \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \left(\int_{\mathbb{R}^d} dy G_0(t, x - y) \right)^p \|u_0\|_p^p < \infty.
\end{aligned} \tag{31}$$

To deal with T_2 , we apply Burkholder's inequality, then Hölder's inequality; we obtain

$$\begin{aligned}
T_2 &\leq C \int_0^T dt \int_{\mathbb{R}^d} dx E \left(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy G(t, x; s, y) h(s, y, u(s, y)) \right. \\
&\quad \times G(t, x; s, y - z) h(s, y - z, u(s, y - z)) \Big)^{\frac{p}{2}} \\
&\leq C \int_0^T dt \int_{\mathbb{R}^d} dx E \left(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy G_0(t - s, y - x) \right. \\
&\quad \times G_0(t - s, y - z - x) |h(s, y, u(s, y))| |h(s, y - z, u(s, y - z))| \Big)^{\frac{p}{2}} \\
&\leq C \int_0^T dt \int_{\mathbb{R}^d} dx \left(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) (G_0(t - s, \cdot - x) * \tilde{G}_0(t - s, \cdot - x))(z) \right)^{\frac{p}{2}-1} \\
&\quad \times \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy G_0(t - s, y - x) G_0(t - s, y - z - x) \\
&\quad \times E \left(|h(s, y, u(s, y))|^{\frac{p}{2}} |h(s, y - z, u(s, y - z))|^{\frac{p}{2}} \right).
\end{aligned} \tag{32}$$

Then, owing to (28),

$$\begin{aligned}
T_2 &\leq C \int_0^T dt \int_{\mathbb{R}^d} dx \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy G_0(t - s, y - x) G_0(t - s, y - z - x) \\
&\quad \times E \left(|h(s, y, u(s, y))|^{\frac{p}{2}} |h(s, y - z, u(s, y - z))|^{\frac{p}{2}} \right).
\end{aligned}$$

Since the covariance functional is translation invariant, the last expression is bounded by

$$\begin{aligned}
&C \int_0^T dt \int_{\mathbb{R}^d} dx \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy G_0(t - s, y) G_0(t - s, y - z) \\
&\quad \times E \left(|h(s, y + x, u(s, y + x))|^{\frac{p}{2}} |h(s, y - z + x, u(s, y - z + x))|^{\frac{p}{2}} \right).
\end{aligned}$$

Applying Fubini's theorem and Schwarz inequality yields

$$\begin{aligned}
T_2 &\leq C \int_0^T dt \int_{\mathbb{R}^d} dx \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy G_0(t-s, y) G_0(t-s, y-z) \\
&\quad \times E(|h(s, y+x, u(s, y+x))|^{\frac{p}{2}} |h(s, y-z+x, u(s, y-z+x))|^{\frac{p}{2}}) \\
&\leq C \int_0^T dt \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy G_0(t-s, y) G_0(t-s, y-z) \\
&\quad \times E\left(\int_{\mathbb{R}^d} dx |h(s, y+x, u(s, y+x))|^p\right)^{\frac{1}{2}} \\
&\quad \times E\left(\int_{\mathbb{R}^d} dx |h(s, y-z+x, u(s, y-z+x))|^p\right)^{\frac{1}{2}} \\
&= C \int_0^T dt \int_0^t ds E\left(\int_{\mathbb{R}^d} dx |h(s, x, u(s, x))|^p\right) \\
&\quad \int_{\mathbb{R}^d} \Gamma(dz) (G_0(t-s, \cdot) * \tilde{G}_0(t-s, \cdot))(z)
\end{aligned} \tag{33}$$

where the last identity holds by the translation invariance of Lebesgue measure.

The Lipschitz continuity of h yields

$$E\left(\int_{\mathbb{R}^d} dx |h(s, x, u(s, x))|^p\right) \leq CE\left(\|u(s, \cdot)\|_p^p + \|h(s, \cdot, 0)\|_p^p\right) \tag{34}$$

Hence, by (28) and (30),

$$\begin{aligned}
T_2 &\leq C_1 \int_0^T dt \int_0^t ds E\left(\|u(s, \cdot)\|_p^p\right) + C_2 \int_0^T dt \int_0^t ds E\left(\|h(s, \cdot, 0)\|_p^p\right) \\
&\leq C_1 \|u\|_{L_p(\Omega \times [0, T]; L_p(\mathbb{R}^d))} + C_3.
\end{aligned} \tag{35}$$

The analysis of T_3 is simpler. Indeed, Hölder's inequality implies

$$\begin{aligned}
T_3 &\leq \int_0^T dt \int_{\mathbb{R}^d} dx \left(\int_0^t ds \int_{\mathbb{R}^d} dy G(t, x; s, y)\right)^{p-1} \\
&\quad \times \int_0^t ds \int_{\mathbb{R}^d} dy G(t, x; s, y) E(|f(s, y, u(s, y))|^p) \\
&\leq C \int_0^T dt \int_{\mathbb{R}^d} dx \int_0^t ds \int_{\mathbb{R}^d} dy G(t, x; s, y) E(|f(s, y, u(s, y))|^p),
\end{aligned}$$

since

$$\sup_{0 \leq s \leq t \leq T, x \in \mathbb{R}^d} \int_{\mathbb{R}^d} dy G(t, x; s, y) < \infty. \tag{36}$$

Then, Fubini's theorem yields

$$\begin{aligned}
T_3 &\leq C \int_0^T dt \int_0^t ds \int_{\mathbb{R}^d} dy E(|f(s, y, u(s, y))|^p) \int_{\mathbb{R}^d} dx G(t, x; s, y) \\
&\leq C \int_0^T dt \int_0^t ds \int_{\mathbb{R}^d} dy E(|f(s, y, u(s, y))|^p).
\end{aligned}$$

The estimate (34) with h replaced by f and (30) imply

$$\begin{aligned} T_3 &\leq C \int_0^T dt \int_0^t ds \left\{ E \left(\|u(s, \cdot)\|_p^p \right) + \|f(s, \cdot, 0)\|_p^p \right\} \\ &\leq C_1 \|u\|_{L_p(\Omega \times [0, T]; L_p(\mathbb{R}^d))} + C_4. \end{aligned} \quad (37)$$

Then, (31), (35) and (37) give

$$\|\mathcal{T}u\|_{L_p(\Omega \times [0, T]; L_p(\mathbb{R}^d))} \leq C_1 \|u\|_{L_p(\Omega \times [0, T]; L_p(\mathbb{R}^d))} + C_2.$$

This completes the proof of Step 1.

Step 2: The mapping \mathcal{T} has a unique fixed point in $L_p(\Omega \times [0, T]; L_p(\mathbb{R}^d))$. Indeed, let $u_1, u_2 \in L_p(\Omega \times [0, T]; L_p(\mathbb{R}^d))$. Proceeding as in Step 1 and by virtue of the Lipschitz property of f and h , we obtain

$$\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{L_p(\Omega \times [0, t]; L_p(\mathbb{R}^d))}^p \leq C_1 \int_0^t ds \|u_1 - u_2\|_{L_p(\Omega \times [0, s]; L_p(\mathbb{R}^d))}^p,$$

for any $0 \leq t \leq T$.

Consequently, for N big enough, the N -th iterate of \mathcal{T} is a contraction on $L_p(\Omega \times [0, T]; L_p(\mathbb{R}^d))$.

■

5 Equivalence between weak and mild formulations

We devote this section to study the relationship between the notions of solution introduced previously, for some particular classes of spde's. As a consequence, we deduce path properties of the mild solution. We start by giving an equivalent weak formulation. Then, we compare the weak and mild formulation when the differential operator is self-adjoint and has nonrandom coefficients.

Let us consider equation (8) written in terms of the sequence $\{W^k, k \geq 0\}$ of independent Brownian motions, that is

$$\begin{aligned} du(t, x) &= [a^{i,j}(t, x)u_{x^i, x^j}(t, x) + b^i(t, x)u_{x^i}(t, x)] \\ &\quad + f(t, x, u(t, x))dt + g^k(t, x, u(t, x))W^k(dt), \end{aligned} \quad (38)$$

$t \in [0, T]$, with initial condition $u(0, \cdot) = u_0$.

We have proved in Theorem 6 the existence of a unique function-valued stochastic process $\{u(t), t \in [0, T]\}$ satisfying

$$\begin{aligned} (u(t, \cdot), \phi) &= (u_0, \phi) + \int_0^t ds (a^{i,j}(s, \cdot)u_{x^i, x^j}(s, \cdot) + (b^i(s, \cdot)u_{x^i}(s, \cdot), \phi) \\ &\quad + \int_0^t ds (f(s, \cdot, u(s, \cdot)), \phi) + \int_0^t W^k(ds) (g^k(s, \cdot, u(s, \cdot)), \phi), \end{aligned} \quad (39)$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, with the pairing (\cdot, \cdot) given in (5). We shall say that the process u is a *weak solution* of equation (39).

The next proposition establishes the equivalence between testing against functions depending on x and functions depending on t and x . To fix the notation, denote by $\mathcal{C}_{t,x;0}^{1,2}$ the space of functions $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of class \mathcal{C}^1 in t , \mathcal{C}^2 in x , with compact support.

Proposition 10 *We assume that the assumptions of Theorem 6 are satisfied. The stochastic process u is a weak solution if and only if for any function $\Phi \in \mathcal{C}_{t,x;0}^{1,2}$, the following identity holds:*

$$\begin{aligned} (u(t, \cdot), \Phi(t, \cdot)) &= (u_0, \Phi(0, \cdot)) + \int_0^t ds (u(s, \cdot), \partial_s \Phi(s, \cdot)) \\ &+ \int_0^t ds (a^{i,j}(s, \cdot) u_{x^i, x^j}(s, \cdot) + b^i(s, \cdot) u_{x^i}(s, \cdot), \Phi(s, \cdot)) \\ &+ \int_0^t ds (f(s, \cdot, u(s, \cdot)), \Phi(s, \cdot)) + \int_0^t W^k(ds) (g^k(s, \cdot, u(s, \cdot)), \Phi(s, \cdot)). \end{aligned} \quad (40)$$

Proof. The “only if” part is trivial. To complete the proof, we proceed into two steps.

Step 1: Let us prove the result in the case where

$$\Phi(t, x) = \varphi(t) \phi(x),$$

with $\varphi \in \mathcal{C}^1([0, T])$ and $\phi \in \mathcal{C}_0^2(\mathbb{R}^d)$.

In equation (39), we set $t = \sigma$, multiply each term by $\varphi'(\sigma)$ and then integrate on $(0, t)$ with respect to σ . We obtain

$$\begin{aligned} &\int_0^t d\sigma \varphi'(\sigma) (u(\sigma, \cdot), \phi) \\ &= \varphi(t) (u_0, \phi) - \varphi(0) (u_0, \phi) \\ &+ \int_0^t d\sigma \varphi'(\sigma) \int_0^\sigma ds (a^{i,j}(s, \cdot) u_{x^i, x^j}(s, \cdot) + b^i(s, \cdot) u_{x^i}(s, \cdot), \phi) \\ &+ \int_0^t d\sigma \varphi'(\sigma) \int_0^\sigma ds (f(s, \cdot, u(s, \cdot)), \phi) \\ &+ \int_0^t d\sigma \varphi'(\sigma) \int_0^\sigma W^k(ds) (g^k(s, \cdot, u(s, \cdot)), \phi) \end{aligned}$$

Set

$$\begin{aligned}
I_1 &= \int_0^t d\sigma \varphi'(\sigma) \int_0^\sigma ds \\
&\quad \times (a^{i,j}(s, \cdot) u_{x^i, x^j}(s, \cdot) + b^i(s, \cdot) u_{x^i}(s, \cdot), \phi), \\
I_2 &= \int_0^t d\sigma \varphi'(\sigma) \int_0^\sigma ds (f(s, \cdot, u(s, \cdot)), \phi), \\
I_3 &= \int_0^t d\sigma \varphi'(\sigma) \int_0^\sigma W^k(ds) (g^k(s, \cdot, u(s, \cdot)), \phi),
\end{aligned}$$

Integrating by parts we obtain,

$$\begin{aligned}
I_1 &= I'_1 - \int_0^t ds \varphi(s) (a^{i,j}(s, \cdot) u_{x^i, x^j}(s, \cdot) + b^i(s, \cdot) u_{x^i}(s, \cdot), \phi), \\
I_2 &= I'_2 - \int_0^t ds \varphi(s) (f(s, \cdot, u(s, \cdot)), \phi), \\
I_3 &= I'_3 - \int_0^t W^k(ds) \varphi(s) (g^k(s, \cdot, u(s, \cdot)), \phi).
\end{aligned}$$

with

$$\begin{aligned}
I'_1 &= \varphi(t) \int_0^t ds (a^{i,j}(s, \cdot) u_{x^i, x^j}(s, \cdot) + b^i(s, \cdot) u_{x^i}(s, \cdot), \phi), \\
I'_2 &= \varphi(t) \int_0^t ds (f(s, \cdot, u(s, \cdot)), \phi), \\
I'_3 &= \varphi(t) \int_0^t W^k(ds) (g^k(s, \cdot, u(s, \cdot)), \phi).
\end{aligned}$$

Thus, noticing that

$$\varphi(t)[(u_0, \phi) + I'_1 + I'_2 + I'_3] = \varphi(t)(u(t, \cdot), \phi),$$

we obtain

$$\begin{aligned}
&\int_0^t d\sigma (u(\sigma, \cdot), \partial_\sigma \Phi(\sigma, \cdot)) = \int_0^t d\sigma \varphi'(\sigma) (u(\sigma, \cdot), \phi) \\
&= \varphi(t)(u(t, \cdot), \phi) - \varphi(0)(u_0, \phi) \\
&\quad - \left\{ \int_0^t d\sigma \varphi(\sigma) (a^{i,j}(\sigma, \cdot) u_{x^i, x^j}(\sigma, \cdot) + b^i(\sigma, \cdot) u_{x^i}(\sigma, \cdot), \phi) \right. \\
&\quad + \int_0^t d\sigma \varphi(\sigma) (f(\sigma, \cdot, u(\sigma, \cdot)), \phi) \\
&\quad \left. + \int_0^t W^k(d\sigma) \varphi(\sigma) (g^k(\sigma, \cdot, u(\sigma, \cdot)), \phi) \right\}.
\end{aligned}$$

yielding (40).

Step 2: The function Φ can be approximated by polynomials

$$p(t, x) = \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} x^\alpha t^\beta,$$

$\alpha = (\alpha_1, \dots, \alpha_d)$, in the norm

$$\|\Phi\| = \sup_{t, x} \left(|\Phi(t, x)| + |\partial_t \Phi(t, x)| + \sum_{|k| \leq 2} |\partial_x^k \Phi(t, x)| \right).$$

(see e.g. Kirillov and Gvishiani [10], pag.77). Let $[0, T] \times K = \text{supp } \Phi$ and p_n be a polynomial defined for $(t, x) \in [0, T] \times K$ such that $\|\Phi - p_n\| < \frac{1}{n}$. We have proved in Step 1 that (40) holds with $\Phi := p_n$. Set $\psi_n(t, x) = p_n(t, x) - \Phi(t, x)$. We now check that the $L^1(\Omega)$ norm of any term in (40), when Φ is replaced by $\psi_n(t, x)$, tends to 0 as n tends to infinity. This shall finish the proof of the proposition.

Indeed, let us first prove that

$$\lim_{n \rightarrow \infty} E(|(u_0, \psi_n(0, \cdot))|) = 0. \quad (41)$$

Assume first $n_0 := 1 - \eta - \frac{2}{p} \geq 0$. Since,

$$\|\psi_n(0, \cdot)\|_{-n_0, q} \leq \|\psi_n(0, \cdot)\|_q \leq \|\psi_n(0, \cdot)\| < \frac{1}{n},$$

for any $q \in (1, \infty)$, Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ yields

$$\begin{aligned} E(|(u_0, \psi_n(0, \cdot))|) &\leq E\left(\int_{\mathbb{R}^d} dx |(1 - \Delta)^{\frac{n_0}{2}} u_0(x)| |(1 - \Delta)^{-\frac{n_0}{2}} \psi_n(0, x)|\right) \\ &\leq E(\|u_0\|_{n_0, p}) \|\psi_n(0, \cdot)\|_{-n_0, q} \\ &\leq C \frac{1}{n}, \end{aligned}$$

yielding (41).

Assume that $n_0 := 1 - \eta - \frac{2}{p} < 0$. The restrictions on η and p yield in this case $-n_0 \in (0, 1)$. Applying the next Lemma 11 to $\psi := \psi_n(0, \cdot)$, we obtain

$$\|\psi_n(0, \cdot)\|_{-n_0, q} \leq C \|\psi_n\|_q < C \frac{1}{n}.$$

Then we can proceed exactly as before and obtain (41).

Set $m_0 = 1 - \eta$. Notice that $m_0 > 0$. Then, as we did before for the case $n_0 \geq 0$,

$$\begin{aligned} E\left(\left|\int_0^t ds (u(s, \cdot), \partial_s \psi_n(s, \cdot))\right|\right) &\leq \int_0^t ds E(\|u(s, \cdot)\|_{m_0, p}) \|\psi_n(s, \cdot)\|_{-m_0, q} \\ &\leq C \frac{1}{n}. \end{aligned} \quad (42)$$

As in the proof of Theorem 6, we set $n = -(1 + \eta)$. From Lemma 5.2 [13], it follows that

$$E\left(\int_0^t ds (\|a^{i,j}(s, \cdot)u_{x^i, x^j}(s, \cdot)\|_{n,p}^p + \|b^i(s, \cdot)u_{x^i}(s, \cdot)\|_{n+1,p}^p)\right) < \infty.$$

Then, since $n + 1 < 0$, by Lemma 11 we obtain

$$\begin{aligned} & E\left(\left|\int_0^t ds (a^{i,j}(s, \cdot)u_{x^i, x^j}(s, \cdot) + b^i(s, \cdot)u_{x^i}(s, \cdot), \psi_n(s, \cdot))\right|\right) \\ & \leq C \frac{1}{n}. \end{aligned} \tag{43}$$

Following (19),

$$\|f(s, \cdot, u(s, \cdot))\|_{n,p} \leq C(\|u\|_p + \|f(s, \cdot, 0)\|_{n,p}).$$

Consequently, by virtue of (14) and Lemma 11

$$\begin{aligned} & E\left(\left|\int_0^t ds (f(s, \cdot, u(s, \cdot)), \psi_n(s, \cdot))\right|\right) \\ & \leq \int_0^t ds E(\|f(s, \cdot, u(s, \cdot))\|_{n,p} \|\psi_n(s, \cdot)\|_{-n,q}) \\ & \leq C \frac{1}{n}. \end{aligned} \tag{44}$$

We now deal with the stochastic integral by considering the $L^2(\Omega)$ -norm. The isometry property of the stochastic integral yields,

$$\begin{aligned} & E\left|\sum_{k=1}^{\infty} \int_0^t dW_s^k (g^k(s, \cdot, u(s, \cdot)), \psi_n(s, \cdot))\right|^2 \\ & = \sum_{k=1}^{\infty} E \int_0^t ds \left(g^k(s, \cdot, u(s, \cdot)), \psi_n(s, \cdot)\right)^2 \end{aligned}$$

Using Schwarz's and Hölder's inequalities, this last expression is bounded by

$$\begin{aligned} & \sup_{s \in [0, T]} \|(1 - \Delta)^{\frac{\eta}{2}} \psi_n(s, \cdot)\|_1 \|(1 - \Delta)^{-\frac{\eta}{2}} \psi_n(s, \cdot)\|_q \\ & \times \int_0^t ds \left\| \left(\sum_{k=1}^{\infty} |(1 - \Delta)^{-\frac{\eta}{2}} g^k(s, \cdot, u(s, \cdot))|^2 \right)^{\frac{1}{2}} \right\|_p^2, \end{aligned}$$

(see [13], pg. 191).

Hence, Lemma 11 and the properties of g imply

$$\begin{aligned} E \left| \sum_{k=1}^{\infty} \int_0^t dW_s^k (g^k(s, \cdot, u(s, \cdot)), \psi_n(s, \cdot)) \right|^2 \\ \leq C \frac{1}{n}. \end{aligned} \quad (45)$$

With the estimates (41)-(45), we finish the proof of the proposition. ■

We conclude the first part of the section giving the auxiliary result that has been used in the proof of the preceding proposition.

Lemma 11 *For any $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\alpha \in (0, 1)$ and $q \in (1, \infty)$,*

$$\|(1 - \Delta)^\alpha \psi\|_q \leq C(\|\Delta \psi\|_q + \|\psi\|_q). \quad (46)$$

Proof. For any $t \in (0, \infty)$, $x \in \mathbb{R}^d$, set $\varphi_t(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp(-\frac{|x|^2}{4t})$, $T_t \psi = \psi * \varphi_t$. It is well-known (see for instance [9]) that

$$(1 - \Delta)^\alpha \psi = C(\alpha) \int_0^\infty dt \frac{e^{-t} T_t \psi - \psi}{t^{\alpha+1}},$$

where $C(\alpha)$ is a positive constant depending only on α .

Fix $t_0 > 1$ and set

$$A_1 = \left\| \int_{t_0}^\infty dt \frac{e^{-t} T_t \psi - \psi}{t^{\alpha+1}} \right\|_q^q.$$

Hölder's inequality yields

$$\begin{aligned} A_1 &\leq \left(\int_{t_0}^\infty dt t^{-\alpha-1} \right)^{q-1} \int_{\mathbb{R}^d} dx \\ &\quad \times \int_{t_0}^\infty dt t^{-\alpha-1} \left(|e^{-t} T_t \psi(x)|^q + |\psi(x)|^q \right). \end{aligned}$$

By Young's inequality,

$$\sup_{t \in [0, T]} \|T_t \psi\|_q \leq \|\psi\|_q \sup_{t \in [0, T]} \|\varphi_t\|_1 = C_0 \|\psi\|_q,$$

with $C_0 = \sup_{t \in [0, T]} \|\varphi_t\|_1^q$. Thus,

$$A_1 \leq C \|\psi\|_q^q,$$

for some positive constant C .

Let us now consider the term

$$A_2 = \left\| \int_0^{t_0} dt \frac{e^{-t} T_t \psi - \psi}{t^{\alpha+1}} \right\|_q^q.$$

Notice that, for any $x \in \mathbb{R}^d$, the mapping $t \mapsto e^{-t}T_t\psi(x)$ is differentiable in $(0, t_0)$ and

$$\begin{aligned}\frac{d}{dt}(e^{-t}T_t\psi(x)) &= e^{-t}(\psi * \Delta\varphi_t - \psi * \varphi_t) \\ &= e^{-t}(\Delta\psi * \varphi_t - \psi * \varphi_t).\end{aligned}$$

Consequently, Young's inequality yields

$$\sup_{t \in (0, T)} \left\| \frac{d}{dt}(e^{-t}T_t\psi) \right\|_q \leq \sup_{t \in [0, T]} \|\varphi_t\|_1 (\|\psi\|_q + \|\Delta\psi\|_q).$$

Therefore, Hölder's inequality implies

$$\begin{aligned}A_2 &\leq \int_{\mathbb{R}^d} dx \left(\int_0^{t_0} \frac{dt}{t^\alpha} \sup_{t \in (0, T)} \left| \frac{d}{dt}(e^{-t}T_t\psi(x)) \right| \right)^q \\ &\leq \int_{\mathbb{R}^d} dx \left(\int_0^{t_0} \frac{dt}{t^\alpha} \sup_{t \in (0, T)} \left| \frac{d}{dt}(e^{-t}T_t\psi(x)) \right|^q \right) \\ &\leq \sup_{t \in [0, T]} \|\varphi_t\|_1^q (\|\psi\|_q^q + \|\Delta\psi\|_q^q) \\ &\leq C(\|\psi\|_q^q + \|\Delta\psi\|_q^q).\end{aligned}$$

Since

$$\|(1 - \Delta)^\alpha \psi\|_q^q \leq C(A_1 + A_2),$$

the upper bounds obtained before for A_1, A_2 give (46) ■

We want now to prove that, if $\{u^W(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ satisfies the weak formulation in the sense of (40) then, it also satisfies the mild formulation. To obtain this result, we restrict the class of operators and assume that \mathcal{L} given in (22) is self-adjoint.

Let us start by proving the following auxiliary result.

Lemma 12 Fix $v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $t \in [0, T]$. Assume that $\text{supp } v \subset K$, for some compact set $K \subset \mathbb{R}^d$. Fix $\varepsilon > 0$ and define $v^t \in \mathcal{L}_{t, x, 0}^{1, 2}$ as follows:

$$v^t(s, x) = \begin{cases} v(x), & \text{if } s = t, x \in \mathbb{R}^d \\ \int_{\mathbb{R}^d} dy v(y) G(t, x; s, y), & \text{if } s < t, x \in \overset{\circ}{K} \\ 0, & \text{if } s \geq t, x \in K(\varepsilon), \end{cases} \quad (47)$$

where $K(\varepsilon) = \{x \in \mathbb{R}^d : d(x, K) > \varepsilon\}$.

Then, with the assumptions of Theorem 6, the following identity holds

$$\begin{aligned}(u^W(t, \cdot), v) &= (u_0, v^t(0, \cdot)) + \int_0^t ds (f(s, \cdot, u^W(s, \cdot)), v^t(s, \cdot)) \\ &\quad + \int_0^t W^k(ds) (g^k(s, \cdot, u^W(s, \cdot)), v^t(s, \cdot)).\end{aligned} \quad (48)$$

Proof. We recall that $G(t, x; s, y)$ is of class \mathcal{C}^1 in the variable t and \mathcal{C}^2 in the variable x . This implies $v^t \in \mathcal{C}_{t,x;0}^{1,2}$. Hence, v^t can be taken as test function in (40) and consequently (48) is equivalent to

$$\begin{aligned} 0 &= \int_0^t ds (u^W(s, \cdot), \partial_s v^t(s, \cdot)) \\ &\quad + \int_0^t ds (a^{i,j}(s, \cdot) u_{x^i, x^j}^W(s, \cdot) + b^i(s, \cdot) u_{x^i}^W(s, \cdot), v^t(s, \cdot)) \\ &= \int_0^t ds (u^W(s, \cdot), \partial_s v^t(s, \cdot)) \\ &\quad + \int_0^t ds (u^W(s, \cdot), \partial_{x, x^j}^2 (a^{i,j}(s, \cdot) v^t(s, \cdot)) - \partial_{x^i} (b^i(s, \cdot) v^t(s, \cdot))). \end{aligned} \quad (49)$$

By the definition of v^t and the properties of the fundamental solution $G(\cdot, \cdot; s, y)$, this is equivalent to

$$0 = \int_{\mathbb{R}^d} dy v(y) \int_0^t ds (u^W(s, \cdot), \mathcal{L}_{s, \cdot}^* G(t, \cdot; s, y)],$$

where \mathcal{L}^* is the adjoint operator of \mathcal{L} (see Friedman [7], pag. 26), that is

$$\mathcal{L}_{t,x}^* u(t, x) = -\frac{\partial}{\partial t} u(t, x) - \sum_{i,j=1}^d \partial_{x^i, x^j}^2 (a^{i,j}(t, x) u(t, x)) + \sum_{i=1}^d \partial_{x^i} (b^i(t, x) u(t, x)).$$

Since \mathcal{L} is a self-adjoint operator, G is symmetric in (x, y) . Consequently $\mathcal{L}_{s, \cdot}^* G(t, \cdot; s, y) = \mathcal{L}_{s, y}^* G(t, y; s, \cdot)$.

By Theorem 15 in Friedman [7], pag. 28, for every fixed $t \in [0, T]$, $y \in \mathbb{R}^d$, we have

$$\mathcal{L}_{s,x}^* G(t, y; s, x) = 0.$$

This finishes the proof of the lemma. ■

Proposition 13 *With the hypotheses of Theorem 6, let $\{u^W(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ be a weak solution in the sense of Proposition 10. Then, for any x -a.e.*

$$\begin{aligned} u^W(t, x) &= (u_0, G(t, x; 0, \cdot)) + \int_0^t ds (f(s, \cdot, u^W(s, \cdot)), G(t, x; s, \cdot)) \\ &\quad + \int_0^t W^k(ds) (g^k(s, \cdot, u^W(s, \cdot)), G(t, x; s, \cdot)). \end{aligned} \quad (50)$$

Proof. For a fixed $t \in [0, T]$, we write the expression (40) with $\Phi(s, x) = v^t(s, x)$, defined in (47). By virtue of Lemma 12 we obtain

$$\begin{aligned} (u^W(t, \cdot), v) &= (u_0, \int_{\mathbb{R}^d} dy v(y) G(t, \cdot; 0, y)) \\ &\quad + \int_0^t ds (f(s, \cdot, u^W(s, \cdot)), \int_{\mathbb{R}^d} dy v(y) G(t, \cdot; s, y)) \\ &\quad + \int_0^t W^k(ds) (g^k(s, \cdot, u^W(s, \cdot)), \int_{\mathbb{R}^d} dy v(y) G(t, \cdot; s, y)). \end{aligned}$$

Fubini's theorem implies

$$\begin{aligned} (u^W(t, \cdot), v) &= \int_{\mathbb{R}^d} dy v(y) [(u_0, G(t, \cdot; 0, y)) \\ &\quad + \int_0^t ds (f(s, \cdot, u^W(s, \cdot)), G(t, \cdot; s, y)) \\ &\quad + \int_0^t W^k(ds) (g^k(s, \cdot, u^W(s, \cdot)), G(t, \cdot; s, y))] \end{aligned}$$

Consequently, for any $x \in K$, a.e. with respect to Lebesgue measure,

$$\begin{aligned} u^W(t, x) &= (u_0, G(t, \cdot; 0, x)) + \int_0^t ds (f(s, \cdot, u^W(s, \cdot)), G(t, \cdot; s, x)) \\ &\quad + \int_0^t W^k(ds) (g^k(s, \cdot, u^W(s, \cdot)), G(t, \cdot; s, x)). \end{aligned}$$

Since $G(t, x; s, y) = G(t, y; s, x)$, for any $s \leq t$ and $x, y \in \mathbb{R}^d$, and K is arbitrary, this is equivalent to (50). Thus, the proof is complete ■

The next result, which is the main conclusion of this section, states that, if there exists a function-valued solution in the weak sense, then it must coincide with the mild solution. We need simultaneously the validity of the assumptions of Theorems 6 and 9. More precisely, we have the following theorem.

Theorem 14 *Suppose that*

1. *The operator \mathcal{L} defined in (22) is self-adjoint and its coefficients are deterministic.*
2. *The functions $a^{i,j}, b^i : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, $i, j = 1, \dots, d$ are $\frac{\alpha}{2}$ -Hölder continuous in $t \in [0, T]$, for some $\alpha \in (0, 1)$ and, for any $t \in [0, T]$, $a^{i,j}(t, \cdot)$ belongs to the space $\mathcal{C}^\beta(\mathbb{R}^d)$, for some $\beta \in (\frac{3}{2}, 2)$, and $b^i(t, \cdot) \in \mathcal{C}^{0,1}(\mathbb{R}^d)$.*
3. *For any $\lambda \in \mathbb{R}^d$, there exist $K, \delta > 0$ such that*

$$\delta |\lambda|^2 \leq a^{i,j}(t, x) \lambda^i \lambda^j \leq K |\lambda|^2.$$

4. *The coefficients of Equation (8) are predictable processes*

$$f, h : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}.$$

For some fixed $p \in [2, \infty)$, the following conditions hold:

- (a) *For any $u, v \in \mathbb{R}$,*

$$\begin{aligned} &\sup_{(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d} \{|f(t, x, u) - f(t, x, v)| + |h(t, x, u) - h(t, x, v)|\} \\ &\leq k |u - v|. \end{aligned}$$

$$(b) \ E\left(\int_0^T ds \|h(s, \cdot, 0)\|_p^p + \|f(s, \cdot, 0)\|_p^p\right) < \infty.$$

5. There exists $\eta \in (\frac{1}{2}, 1)$ such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^2)^\eta} < \infty.$$

6. $u_0 \in L_p(\mathbb{R}^d)$.

Then $u^W = u^M$ as processes in $L_p(\Omega \times [0, T]; L_p(\mathbb{R}^d))$. Consequently, ω -a.s., $u^W(t, x) = u^M(t, x)$, a.e. with respect to Lebesgue measure on $[0, T] \times \mathbb{R}^d$.

Remark 15 1. The above hypothesis 5 implies that $\|R_{\eta, d}\|_{\mathcal{H}} < \infty$.

Under this condition, we have proved in Lemma 3, that $\|\bar{h}\|_p \leq C\|h\|_p$, where $\bar{h}(x) = \|R_{\eta, d}(x - \cdot)h\|_{\mathcal{H}}$. Therefore the assumption 4 (b) implies

$$E \int_0^T ds \|\bar{h}(s, \cdot, 0)\|_p^p < \infty,$$

(see (14) in Theorem 6).

2. From the relation $\|\cdot\|_{n, p} \leq \|\cdot\|_{m, p}$, $n \leq m$, it follows trivially that $\|\cdot\|_{-1-\eta, p} \leq \|\cdot\|_p$. Thus, assumption 4 (b) implies

$$E \int_0^T ds \|f(s, \cdot, 0)\|_{-1-\eta, p}^p < \infty,$$

(see again (14) in Theorem 6).

3. The above remarks show that the assumptions of Theorem 6 and of Theorem 9 are fulfilled. Hence, the existence of u^W satisfying the weak formulation of equation (8) and u^M satisfying the mild formulation is assured.

Notice that

$$\mathcal{H}_p^{1-\eta}(T) \subset L_p(\Omega \times [0, T]; L_p(\mathbb{R}^d)).$$

Proof of theorem 14 Equation (50) can be now written as

$$\begin{aligned} u^W(t, x) &= \int_{\mathbb{R}^d} dy u_0(y) G(t, x; 0, y) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} f(s, y, u^W(s, y)) G(t, x; s, y) \\ &+ \int_0^t W^k ds \int_{\mathbb{R}^d} g^k(s, y, u^W(s, y)) G(t, x; s, y). \end{aligned}$$

We next prove that

$$E \left(\int_0^T dt \|u^W(t, \cdot) - u^M(t, \cdot)\|_p^p \right) = 0. \quad (51)$$

Indeed, from the equations satisfied by u^w and u^M , respectively, it follows that

$$E \left(\int_0^T dt \|u^W(t, \cdot) - u^M(t, \cdot)\|_p^p \right) \leq C(R_1 + R_2),$$

with

$$R_1 = E \int_0^T dt \int_{\mathbb{R}^d} dx \left| \int_0^t ds \int_{\mathbb{R}^d} dy (f(s, y, u^W(t, y)) - f(s, y, u^M(t, y))) \times G(t, x; s, y) \right|^p,$$

$$R_2 = E \int_0^T dt \int_{\mathbb{R}^d} dx \left| \int_0^t W^k(ds) \int_{\mathbb{R}^d} dy (g^k(s, y, u^W(t, y)) - g^k(s, y, u^M(t, y))) \times G(t, x; s, y) \right|^p.$$

Following the arguments of the proof of Theorem 9 and by virtue of the Lipschitz assumptions on f and h , we obtain

$$R_1 + R_2 \leq C \int_0^T dt \int_0^t ds E \left(\int_{\mathbb{R}^d} dx |u^W(s, x) - u^M(s, x)|^p \right).$$

We conclude by Gronwall's lemma applied to the function

$$\Psi(T) = E \left(\int_0^T dt \|u^W(t, \cdot) - u^M(t, \cdot)\|_p^p \right)$$

for $T \geq 0$.

The conclusion of Theorem 14 can be strengthened assuming, for instance, instead of 4(b), that $f(t, x, 0) = h(t, x, 0) = 0$ and $u_0 \in \cap_{p \geq 2} L_p(\mathbb{R}^d)$. Indeed, in this case it has been proved that a.s.

$$u^W \in \mathcal{C}^{\gamma_1, \gamma_2}([0, T] \times \mathbb{R}^d),$$

with $\gamma_1 < \frac{1-\eta}{2}$, $\gamma_2 < 1 - \eta$, and we can show that ω -a.s. u^M owns the same property.

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